Math 222A Lecture 17 Notes

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1 Using the Fourier Transform to Find Fundamental Solutions

1.1 The Paley-Wiener theorem and the Fourier transform of even and odd functions

We have been looking at the Fourier transform

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} u(x)\, dx.$$

We initially defined $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, but we can also define it $L^2 \to L^2$ (with the isometry property) and $\mathcal{S}' \to \mathcal{S}'$. We have also seen that $\mathcal{F} : L^1 \to L^\infty$.

Last time, we also saw that

$$\widehat{H} = \frac{i}{x - i0}.$$

If $u \in \mathcal{S}'$ with $\operatorname{supp} u \subseteq [0, \infty)$, then $\widehat{u}$ has a holomorphic extension to $\{\operatorname{Im} z \leq 0\}$. If u is a measure, then $\widehat{u}$ is bounded in $\{\operatorname{Im} z \leq 0\}$. This leads us to the following property. First, let's generalize this statement.

Suppose $\operatorname{supp} u \subseteq [a, \infty)$. Then

$$\widehat{u}(\xi + i\zeta) = \int e^{ix\xi + x\zeta} u(x)\, dx,$$

so

$$|\widehat{u}(\xi + i\zeta)| \leq e^{a\zeta}.$$

The best we can hope for is a bound of the form $e^{a\zeta}|\xi|^N$.

Theorem 1.1 (Paley-Wiener)**.** *$u \in \mathcal{S}'$ has $\operatorname{supp} u \subseteq [a, \infty)$ if and only if $\widehat{u}$ has a holomorphic extension to the lower half-plane such that*

$$|\widehat{u}(z)| \leq e^{-a \operatorname{Im} z} |z|^N.$$

Remark 1.1. There is a Paley-Wiener theorem in higher dimensions. If $\operatorname{supp} u \subseteq K$ for some compact K, then $\widehat{u}(\xi)$ is defined for $\xi \in \mathbb{C}^n$. Instead of getting the support of u as K in the other direction, we get the convex hull of K.

We can also think of the $e^{-}ix \cdot \xi$ in the Fourier transform as $\cos(-x \cdot \xi) + i \sin(-x \cdot \xi)$.

- If u is real and even, hen $\widehat{u}$ is real and even.
- If u is real and odd, then $\widehat{u}$ is imaginary and odd.
- If u is imaginary and even, then $\widehat{u}$ is imaginary and even.
- If u is imaginary and odd, then $\widehat{u}$ is real and odd.

1.2 Using the Fourier transform to find fundamental solutions

Suppose we have a constant coefficient partial differential operator $P(\partial)$, and we want to compute a fundamental solution $P(\partial)K = \delta_0$. Let $D = \frac{1}{i}\partial$. Taking the Fourier transform gives

$$P(\xi)\widehat{K} = \frac{1}{(2\pi)^{n/2}}\mathbf{1}.$$

This tells us that

$$\widehat{K} = \frac{1}{(2\pi)^{n/2}}P(\xi).$$

So we can invert the Fourier transform to get K:

$$K = \frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left(\frac{1}{P(\xi)}\right).$$

Here are some issues.

- $p(\xi)$ may have zeros.
- If p has zeroes, then $\frac{1}{p}$ is not uniquely determined as a distribution.
- This procedure only gives fundamental solutions which are temperate distributions.

The easy case is when $p(\xi) \neq 0$ for any $\xi \in \mathbb{R}^n$. Then $\frac{1}{p} \in \mathcal{S}'$, so this computation is justified.

Example 1.1. Suppose $P = -\partial_x^2 + 1 = D_x^2 + 1$. Then $P(\xi) = (1 + \xi^2)$. So we compute

$$K(x) = \mathcal{F}^{-1}\left(\frac{1}{1+\xi^2}\right).$$

This $K(x)$ is real and even. We are looking at

$$\int_{\mathbb{R}} \frac{1}{\xi^2} e^{ix\xi}\, d\xi.$$

This integrand has a pole at i and a pole at $-i$. However, we can expend this using partial fractions:

$$\frac{1}{1+\xi^2} = \frac{i}{2}\frac{1}{\xi+i} - \frac{i}{2}\frac{1}{\xi-i},$$

where the first term is holomorphic if $\operatorname{Im}\zeta > 0$ and the second is holomorphic if $\operatorname{Im}\zeta < 0$. So the Paley-Wiener theorem tells us that the first one will have an inverse Fourier transform supported in $(-\infty, 0]$, and the second one will have an inverse Fourier transform supported in $[0, \infty)$.

If $x < 0$, we can use complex analysis to say

$$\int_{\mathbb{R}} \frac{1}{\xi+i} e^{ix\xi}\, d\xi = \text{Residue at } i = e^x.$$

A similar computation for $x > 0$ suggests that we should get

$$\int_{\mathbb{R}} \frac{1}{\xi^2} e^{ix\xi}\, d\xi = ce^{-|x|}.$$

In general, if K is a fundamental solution, then so will be $K + K_0$, where K_0 solves the homogeneous equation $P(\partial)K_0 = 0$. In this case, our general solution is $K = ce^{|x|} + c_1 e^x + c_2 e^{-x}$. We did not get these latter two terms before because they are not temperate distributions.

Example 1.2. If $P = -\Delta + 1$, then $P(\xi) = \xi^2 + 1$ in $\mathbb{R}^n$. Then

$$K = \mathcal{F}^{-1}\left(\frac{1}{1+\xi^2}\right)$$

gives the unique temperate fundamental solution. Note that $e^{ix\cdot\xi}$ is a solution iff $1+\xi^2 = 0$. In 3 dimensions, this is $K(x) = e^{-|x|}\frac{1}{|x|}$.

Example 1.3. Let $P = -\Delta$, so $P(\xi) = \xi^2$. Then $K = \frac{1}{\xi^2}$ is locally integrable in $\mathbb{R}^n$ if $n \geq 3$. So if $n \geq 3$, we get that $K \in \mathcal{S}'$ is a homogeneous temperate distribution. Since $\frac{1}{\xi^2}$ is homogeneous of order -2, $K = \mathcal{F}^{-1}(\frac{1}{\xi^2})$ will be homogeneous of order $2 - n$.

Proposition 1.1. *If u is homogeneous of order s, then $\widehat{u}$ is homogeneous of order $-n-s$.*

The example to keep in mind to make sure your numbers are right is $\widehat{\delta} = \frac{1}{(2\pi)^{n/2}}$. The Dirac mass is homogeneous of order $-n$, whereas this constant function is homogeneous of order 0.

Example 1.4. If $P = -\Delta$ with $n = 2$, perform the same computation as before, but interpret $\frac{1}{\xi^2}$ as a distribution:

$$\frac{1}{|\xi|^2}(\varphi) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B(0,\varepsilon)} \frac{\varphi(\xi)}{|\xi|^2} \, d\xi - \varphi(0) \ln \varepsilon,$$

so we pay a price of log, which makes us lose the homogeneity property.

Example 1.5. Suppose $P(\xi) = A\xi \cdot \xi$, where A is a positive deifnite matrix. This is a second order, elliptic, constant coefficient PDE with $P = a^{i,j}\partial_i \partial_j$. We can transform $A \to \mathrm{Id}$ by a linear transformation. Let $x = By$, so $x \cdot \xi = By \cdot \xi = y \cdot B^\top \xi$. If we carry out the computation, we end up with

$$K = \frac{1}{(A^{-1}x \cdot x)^{(n-2)/2}}.$$

Hormander's book extensively discusses how the Fourier transform behaves under linear changes of coordinates.

1.3 Fundamental solution of the heat equation

Recall the heat equation

$$(\partial_t - \Delta)u = f.$$

We think of u as the temperature of an infinite solid and f as describing the heat sources. This is also called the *diffusion equation*, since we can, for example, interpret $u(t, x)$ as a local concentration of salt in the water of an ocean. In probability theory, the heat equation has connections to Brownian motion, where we let a particle move randomly at every time, independently of the movement at other times.

Our Fourier variables will be ξ (corresponding to x) and τ (corresponding to t). We can write our operator as[1]

$$\partial_t - \Delta = iD_t + D_x^2,$$

so

$$P(\xi, \tau) = iT + \xi^2,$$

which vanishes only at $\tau = 0, \xi = 0$. Is $\frac{1}{i\tau + \xi^2} \in L^1_{\text{loc}}$? Yes! The $1/\tau$ increases the local integrability of this expression, so we will not need to make a distinction between the cases $n = 2$ and $n \geq 3$. We want to calculate

$$\mathcal{F}^{-1}\left(\frac{1}{i\tau + \xi^2}\right).$$

[1] Warning: Evans' book means something different with the D notation.

First integrate in τ: We have a pole at $\tau = i\xi^2$. This pole is in the upper half plane, so $\mathcal{F}_\tau^{-1}(\frac{1}{i\tau+\xi^2})$ is supported where $t > 0$. This says that the evolution of heat is well-defined in the future, rather than in the past. We conclude that

$$\mathcal{F}_\tau^{-1}\left(\frac{1}{i\tau + \xi^2}\right) = ce^{-t\xi^2}\mathbb{1}_{\{t\geq 0\}}.$$

for some constant c. Then we can calculate

$$\mathcal{F}^{-1}\left(\frac{1}{i\tau + \xi^2}\right) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{x^2}{4t}}\mathbb{1}_{\{t\geq 0\}}.$$

Here is another approach. We can try to solve

$$\begin{cases} (\partial_t - \Delta)u = 0 \\ u(0) = \delta_0 \end{cases}$$

Take the Fourier transform in x to get

$$\begin{cases} (\partial_t + \xi^2)\widehat{u} = 0 \\ \widehat{u}(0) = \frac{1}{(2\pi)^{n/2}}. \end{cases}$$

This gives

$$\widehat{u} = \frac{1}{(2\pi)^{n/2}}e^{-t\xi^2}.$$

So we get the same result.

For $t > 0$, we can consider

$$\begin{cases} (\partial_t - \Delta)u = 0 \\ u(0) = u_0. \end{cases}$$

Extend u to

$$\widetilde{u} = \begin{cases} u & t > 0 \\ 0 & y < 0. \end{cases}$$

Then

$$(\partial_t - \Delta)\widetilde{u} = u_0(x)\delta_{t=0}.$$

Here, $u_0 = \delta_{x=0}$, so $u_0\delta_{t=0} = \delta_{(0,0)}$.